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Cramér–Rao Inequalities for Operator-Valued Measures in Quantum Mechanics

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Abstract

The theory of estimation of parameters of quantum-mechanical density operators is expressed in terms of the measurement of operator-valued measures. Lower bounds on mean-square errors of parameter estimates are set by two quantum-mechanical forms of the Cramér–Rao inequality of classical statistics, derived here in terms of such measures. The results are exemplified by the simultaneous estimation of the real and imaginary parts of the complex amplitude of a coherent oscillation in the presence of thermal noise.

1. *Quantum Estimation Theory*

Estimation theory in its classical statistical form starts with sets of observations or data $(x_1, x_2, \ldots, x_n) = \mathbf{x}$ described by a probability density function (p.d.f.) $p(\mathbf{x}|\boldsymbol{\theta})$ depending on certain unknown parameters $(\theta_1, \theta_2, \ldots, \theta_m) = \boldsymbol{\theta}$. From a given set of data $\mathbf{x}$ one is to deduce the most appropriate values of the parameters $\boldsymbol{\theta}$; the values one arrives at are called *estimates* of $\boldsymbol{\theta}$ and are designated by $(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m) = \hat{\boldsymbol{\theta}}$. In determining the location of a star from its image on a photographic plate, for example, the x_i's might be the densities of developed grains at certain points of the plate, and θ_1 and θ_2 would be the x- and y-coordinates of the center of the stellar image. The p.d.f. $p(\mathbf{x}|\boldsymbol{\theta})$ would embody the illuminance distribution of the image and the 'statistics' of grain formation as it depends on the illuminance at each point.

The estimates are obtained by certain mathematical operations on the data, represented by the functions, or *estimators*, $\hat{\theta}_i(\mathbf{x})$, $i = 1, 2, \ldots, m$. In its most general form, estimation theory treats the parameters $\boldsymbol{\theta}$ as random variables with a certain prior p.d.f. $z(\boldsymbol{\theta})$. Recognizing that the estimates $\hat{\theta}$

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will seldom equal the true values $\boldsymbol{\theta}$ of the parameters, it postulates a cost function $C(\hat{\boldsymbol{\theta}},\boldsymbol{\theta})$ measuring the seriousness of the discrepancies. A cost function commonly used because of its mathematical tractability is the weighted sum of squared errors, the quadratic cost function

$$C(\hat{\boldsymbol{\theta}},\boldsymbol{\theta})=\sum_{i=1}^{m} g_i(\hat{\theta}_i-\theta_i)^2, \qquad g_i>0 \tag{1.1}$$

or more generally

$$C(\hat{\boldsymbol{\theta}},\boldsymbol{\theta})=\sum_{i=1}^{m}\sum_{j=1}^{m} g_{ij}(\hat{\theta}_i-\theta_i)(\hat{\theta}_j-\theta_j) \tag{1.2}$$

where $\mathbf{G}=\|g_{ij}\|$ is a positive-definite $m\times m$ matrix. Whatever the cost function, estimators $\hat{\boldsymbol{\theta}}(\mathbf{x})$ are sought for which the average cost

$$\bar{C}=\int_{\mathfrak{X}}\int_{\mathfrak{P}} z(\boldsymbol{\theta})\,C(\hat{\boldsymbol{\theta}}(\mathbf{x}),\boldsymbol{\theta})\,p(\mathbf{x}|\boldsymbol{\theta})\,d^n\mathbf{x}\,d^m\boldsymbol{\theta} \tag{1.3}$$

is minimum (Wald, 1939; Blackwell & Girshick, 1954). Here $\mathfrak{P}$ denotes the space of the parameters $\boldsymbol{\theta}$ and their estimates $\hat{\boldsymbol{\theta}}$, $\mathfrak{X}$ the space of the data $\mathbf{x}$; $d^m\boldsymbol{\theta}$ and $d^n\mathbf{x}$ are elements of integration in those spaces.

When the observed entity, or system S, must be treated quantum-mechanically, it is described by a density operator $\rho(\boldsymbol{\theta})$ that depends on the unknown parameters, or *estimanda*, $\boldsymbol{\theta}$. Suppose, for example, that the position q_0 and the momentum p_0 of a free particle are to be estimated at a certain time t, and it is known that the particle has been prepared as a minimum-uncertainty Gaussian wavepacket. Then $\boldsymbol{\theta}=(q_0,p_0)$, and the density operator is

$$\rho(q_0,p_0)=|\alpha(q_0,p_0)\rangle\langle\alpha(q_0,p_0)| \tag{1.4}$$

where $|\alpha\rangle$ is a coherent state as defined by Glauber (1963) and

$$\alpha(q_0,p_0)=\frac{q_0}{2\Delta q}+i\frac{p_0}{2\Delta p}, \qquad \Delta q\,\Delta p=\hbar/2 \tag{1.5}$$

Δq and Δp being the r.m.s. uncertainties in position and momentum.

If the measurements $\mathbf{x}$ to be made on the system are specified, the joint p.d.f. $p(\mathbf{x}|\boldsymbol{\theta})$ of their outcomes can in principle be calculated from $\rho(\boldsymbol{\theta})$, and the best manner of processing them in order to estimate the parameters $\boldsymbol{\theta}$ can be sought through the classical theory. There is, however, a variety of measurements that might be made on the system S, and quantum estimation theory not only seeks the best estimators $\hat{\boldsymbol{\theta}}(\mathbf{x})$ based on the outcomes $\mathbf{x}$ of a set of measurements, but asks in the first place what are the best measurements to make (Helstrom, 1972, Section 6).

In order to carry out this task it is necessary to express the average cost $\bar{C}$ not as in equation (1.3) in terms of the outcomes of certain measurements, but in a form permitting the greatest arbitrariness compatible with the laws of quantum mechanics. When a single parameter θ is to be estimated on the basis of a single observation, we can suppose that a quantum-mechanical operator Θ with eigenvalues $\hat{\theta}$ is being measured, and we can write the average cost as

$$\bar{C} = \text{Tr} \int z(\theta) C(\Theta, \theta) \rho(\theta) \, d\theta \tag{1.6}$$

where $C(\Theta, \theta)$ is an operator obtained by replacing the estimate $\hat{\theta}$ in the cost function $C(\hat{\theta}, \theta)$ by the operator Θ. For a quadratic cost function, $(\hat{\theta} - \theta)^2$, Personick (1971b) has derived an operator equation from which the optimum operator Θ can be determined. Yuen & Lax (1972) have extended this to estimates of two parameters combined into a single complex parameter, with the cost assessed by the sum of the squared errors in each. Solving the necessary operator equations in these cases is usually very difficult, and for estimation of more than one parameter with an arbitrary cost function $C(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})$ the minimization of the expected cost $\bar{C}$ has not, to the writer's knowledge, been explicitly achieved. In the next section we shall show how the average cost $\bar{C}$ can be expressed in a general formulation of quantum measurement in terms of so-called operator-valued measures. Even classical statistical estimation theory leads in general to complicated procedures whose minimum cost is not easily determined.

The impediments to finding and assessing the optimum estimators have led theorists to resort to setting lower bounds on average error costs of procedures for estimating a given set $\hat{\boldsymbol{\theta}}$ of parameters. For quadratic cost functions these bounds are provided by the Cramér–Rao inequality, which was developed for estimates of parameters of classical probability density functions (Cramér, 1946; Rao, 1945). Quantum-mechanical counterparts have been developed by Helstrom (1967a, 1968a) and, for pairs of parameters combined into complex numbers, by Yuen & Lax (1972). Their bound has been generalized to apply to a number of arbitrary real parameters by Helstrom & Kennedy (1972). Our aim here is to reformulate that work in terms of operator-valued measures and thus to provide a more concise derivation of the basic inequalities. These are of interest to physicists because they quantify fundamental limitations imposed by nature on the accuracy with which physical quantities can be estimated.

2. *Operator-Valued Measures*

If the measurements to be made on the system S are described by commuting operators X_j acting in its Hilbert space $\mathscr{H}_S$ and having eigenvalues $x_j, j = 1, 2, \ldots, m$, the estimators $\hat{\theta}_j(\mathbf{x})$ can be replaced by commuting operators $\Theta_j = \hat{\theta}_j(X_1, X_2, \ldots, X_m)$, and we can envision measuring not the X_j's, but the Θ_j's on S. Let their simultaneous eigenstates be

$$|\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m\rangle = |\hat{\boldsymbol{\theta}}\rangle$$

and their eigenvalues be $\hat{\theta}_j$,

$$\Theta_j |\hat{\boldsymbol{\theta}}\rangle = \hat{\theta}_j |\hat{\boldsymbol{\theta}}\rangle \tag{2.1}$$

Then the outcomes of measuring the operators Θ_j will be the estimates $\hat{\theta}_j$ of the parameters θ_j, the joint conditional p.d.f. of the estimates will be

$$p(\hat{\boldsymbol{\theta}}|\boldsymbol{\theta}) = \langle \hat{\boldsymbol{\theta}} | \rho(\boldsymbol{\theta}) | \hat{\boldsymbol{\theta}} \rangle \tag{2.2}$$

when the true values of the parameters are $\boldsymbol{\theta}$ and the density operator is $\rho(\boldsymbol{\theta})$, and the expected cost will be, in place of equation (1.3),

$$\bar{C} = \mathrm{Tr}_S \int_{\mathfrak{P}} z(\boldsymbol{\theta})\, C(\boldsymbol{\Theta}, \boldsymbol{\theta})\, \rho(\boldsymbol{\theta})\, d^m \boldsymbol{\theta} \tag{2.3}$$

where the operator $C(\boldsymbol{\Theta},\boldsymbol{\theta})$ is obtained from the cost function $C(\hat{\boldsymbol{\theta}},\boldsymbol{\theta})$ by replacing the estimates $\hat{\boldsymbol{\theta}}$ by the operators $\Theta_1, \Theta_2, \ldots, \Theta_m$, and where Tr_S stands for a trace over the Hilbert space $\mathscr{H}_S$.

A more general formulation of quantum measurement is expressed in terms of an operator-valued measure. Consider the parameter space $\mathfrak{P}$ to be divided arbitrarily into disjoint regions Δ_k, to each of which is assigned a positive-definite Hermitian operator $X(\Delta_k)$ acting in $\mathscr{H}_S$ and having the following properties:

(a) $$X(\varnothing) = \mathbf{0} \tag{2.4}$$

where $\varnothing$ is the empty set in $\mathfrak{P}$, $\mathbf{0}$ the zero operator in $\mathscr{H}_S$;

(b) $$\sum_{i \in I} X(\Delta_i) = X\Big(\sum_{i \in I} \Delta_i\Big), \qquad \Delta_1 \cap \Delta_2 \cap \ldots \cap \Delta_k \cap \ldots = \varnothing \tag{2.5}$$

where I is any set of the indices i (additivity);

(c) $$X(\mathfrak{P}) = \mathbf{1} \tag{2.6}$$

where $\mathbf{1}$ is the identity operator in $\mathscr{H}_S$. The operators $X(\Delta_i)$ need not commute. We think of the parameter space $\mathfrak{P}$ as divisible into infinitesimal regions $d^m\boldsymbol{\theta}'$ at all points $\boldsymbol{\theta}'$, with each of which is associated the infinitesimal positive-definite operator $X(\boldsymbol{\theta}'; d^m\boldsymbol{\theta}')$ in such a way that for any finite region Δ,

$$X(\Delta) = \int_{\Delta} X(\boldsymbol{\theta}'; d^m \boldsymbol{\theta}') \tag{2.7}$$

An example will be given in Section 3. The collection $\{X(\Delta)\}$ of operators so generated and obeying equations (2.4–2.6) makes up the operator-valued measure. When we speak of 'measuring' $\{X(\Delta)\}$ we mean applying to the system a measuring device that registers a set of outcomes $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m)$ lying in the region Δ of $\mathfrak{P}$ with probability

$$\Pr\{\hat{\boldsymbol{\theta}} \in \Delta | \boldsymbol{\theta}\} = \mathrm{Tr}_S[\rho(\boldsymbol{\theta})\, X(\Delta)] \tag{2.8}$$

where $\rho(\boldsymbol{\theta})$ is the density operator of the system.

The commuting operators Θ_j defined in equation (2.1) provide a special kind of operator-valued measure known as a 'projection-valued measure', with

$$X(\hat{\boldsymbol{\theta}}; d^m \hat{\boldsymbol{\theta}}) = |\hat{\boldsymbol{\theta}}\rangle \langle \hat{\boldsymbol{\theta}}|\, d^m \hat{\boldsymbol{\theta}};$$

the operators

$$X(\Delta) = \int_{\Delta} |\hat{\theta}\rangle\langle\hat{\theta}|\, d^m\hat{\theta} \tag{2.9}$$

are projection operators in $\mathscr{H}_S$,

$$X(\Delta_j)\, X(\Delta_k) = X(\Delta_j)\, \delta_{jk}, \qquad \Delta_j \cap \Delta_k = \varnothing$$

When quantum measurements are formulated thus, the joint conditional p.d.f. of the estimates $\hat{\theta}$ needed for evaluating the expected cost as in equation (1.3) is given by

$$p(\hat{\theta}|\theta)\, d^m\hat{\theta} = \mathrm{Tr}_S\, \rho(\theta)\, X(\hat{\theta}; d^m\hat{\theta}) \tag{2.10}$$

and the average cost can be written

$$\bar{C} = \mathrm{Tr}_S \int_{\mathfrak{P}} \int_{\mathfrak{P}} z(\theta)\, C(\hat{\theta}, \theta)\, \rho(\theta)\, X(\hat{\theta}; d^m\hat{\theta})\, d^m\theta \tag{2.11}$$

the minimization of which has been discussed by Holevo (1972).

The need for this more general formulation of quantum measurement has been indicated in two ways. First of all, one can imagine adjoining to the system S an auxiliary system, or *apparatus* A, and measuring commuting estimators $\Theta_1, \Theta_2, \ldots, \Theta_m$ on one or both of them after they have interacted. Because the expected Bayes cost $\bar{C}$ is independent of the time at which the measurement is made, it will be the same as if commuting estimators were measured on S and A before their interaction (Helstrom & Kennedy, 1972). At that time the density operator for the combination would have the form $\rho(\theta) \otimes \rho_A$ in the product space $\mathscr{H}_S \otimes \mathscr{H}_A$, where the density operator ρ_A of the apparatus is independent of the unknown parameters θ and acts in the Hilbert space $\mathscr{H}_A$ of the apparatus. The conditional probability that the estimates $\hat{\theta}$ lie in an arbitrary region of the parameter space $\mathfrak{P}$ is now

$$\Pr\{\Delta|\theta\} = \mathrm{Tr}_{S+A}\, \rho_S(\theta) \otimes \rho_A \int_{\Delta} |\hat{\theta}\rangle\langle\hat{\theta}|\, d^m\hat{\theta} \tag{2.12}$$

where the states $|\hat{\theta}\rangle$ are defined in the product Hilbert space $\mathscr{H}_S \otimes \mathscr{H}_A$, over which the trace Tr_{S+A} is taken. This probability can be written

$$\Pr\{\Delta|\theta\} = \mathrm{Tr}_S\, \rho_S(\theta)\, X(\Delta) \tag{2.13}$$

where

$$X(\Delta) = \mathrm{Tr}_A\, \rho_A \int_{\Delta} |\hat{\theta}\rangle\langle\hat{\theta}|\, d^m\hat{\theta} \tag{2.14}$$

denotes an operator-valued measure in $\mathscr{H}_S$ as defined in equations (2.4–2.6). It is not, however, in general a projection-valued measure in $\mathscr{H}_S$.

A second reason for bringing in operator-valued measures is the possibility of making discrete measurements on the system S at a succession of times $t_1, t_2, t_3, \ldots$, between which the system evolves as described by the Schrödinger equation. The outcome of all these discrete measurements, which could be as fine and as extensive as desired or necessary, would constitute data from which estimates of the parameters $\boldsymbol{\theta}$ might be derived. Benioff (1972a, b) has shown that such a succession of discrete measurements is equivalent to measuring on the system S an operator-valued measure, even when they are made at an infinite sequence of times t_1, t_2, $t_3, \ldots$, and when decisions about what to measure next are, after some or all of the measurements, made on the basis of the previous findings.

According to a theorem of M. A. Neumark's (1943), a noncommuting operator-valued measure $\{X(\Delta)\}$ can be extended to a commuting, projection-valued measure $\{E(\Delta)\}$ by embedding the defining Hilbert space $\mathscr{H}_S$ in a larger Hilbert space $\mathscr{H}'$, in which

$$X(\Delta) = PE(\Delta)P \tag{2.15}$$

where P is a projection operator projecting arbitrary vectors in $\mathscr{H}'$ into the subspace corresponding to $\mathscr{H}_S$, and each $E(\Delta)$ is a projection operator in $\mathscr{H}'$. The operators Θ_j' defined by

$$\Theta_j' = \int_{\mathfrak{P}} \theta_j' \, E(\boldsymbol{\theta}'; d^m \boldsymbol{\theta}') \tag{2.16}$$

where $X(\boldsymbol{\theta}'; d^m\boldsymbol{\theta}') = PE(\boldsymbol{\theta}'; d^m\boldsymbol{\theta}')P$, will then be commuting estimators measurable in $\mathscr{H}'$ in the conventional quantum-mechanical sense. The generalization of the concept of quantum measurement to the operator-valued measures defined in equations (2.4–2.6) requires no break with accepted principles of quantum mechanics.

How the extended Hilbert space $\mathscr{H}'$ is constructed has been described by Achieser & Glasmann (1960). With each state vector $|\psi\rangle$ in $\mathscr{H}_S$ we form all possible combinations $|\Delta, \psi\rangle$ with regions Δ of $\mathfrak{P}$; these are vectors in $\mathscr{H}'$. In the space $\mathscr{H}'$ the scalar product, indicated by a prime, of two such 'vectors' $|\Delta_1, \psi_1\rangle$ and $|\Delta_2, \psi_2\rangle$ is defined by

$$\langle \Delta_2, \psi_2 | \Delta_1, \psi_1 \rangle' = \langle \psi_2 | X(\Delta_1 \cap \Delta_2) | \psi_1 \rangle \tag{2.17}$$

where the expression on the right is the usual Dirac bracket expression. This new scalar product $\langle \cdot | \cdot \rangle'$ has all the properties of an ordinary scalar product. The vectors $|\mathfrak{P}, \psi\rangle$ span a subspace of $\mathscr{H}'$ that is identified with the original Hilbert space $\mathscr{H}_S$; indeed,

$$\langle \mathfrak{P}, \psi_2 | \mathfrak{P}, \psi_1 \rangle' = \langle \psi_2 | \psi_1 \rangle \tag{2.18}$$

because of equations (2.17) and (2.6). The projection of an arbitrary vector $|\Delta, \psi\rangle$ onto $\mathscr{H}_S$ is the vector $|\mathfrak{P}, X(\Delta)\psi\rangle$, and the operator that everywhere effects this is called P,

$$P|\Delta, \psi\rangle = |\mathfrak{P}, X(\Delta)\psi\rangle \tag{2.19}$$

The operator $E(\Delta)$ in $\mathscr{H}'$ corresponding to $X(\Delta)$ in $\mathscr{H}_S$ through equation (2.15) is defined for each region Δ' of $\mathfrak{P}$ and each state $|\psi\rangle$ of $\mathscr{H}_S$ by

$$E(\Delta)|\Delta', \psi\rangle = |\Delta \cap \Delta', \psi\rangle; \tag{2.20}$$

the set $\{E(\Delta)\}$ of operators so defined is shown by Achieser & Glasmann (1960) to represent a projection-valued measure, with $E(\mathfrak{P}) = \mathbf{1}'$, the identity operator in $\mathscr{H}'$.

If the density operator $\rho(\boldsymbol{\theta})$ is expressed in terms of its eigenvalues P_k and its eigenstates $|\varphi_k\rangle$ in $\mathscr{H}_S$,

$$\rho(\boldsymbol{\theta}) = \sum_k P_k |\varphi_k\rangle\langle\varphi_k| \tag{2.21}$$

we can define the density operator in the extended Hilbert space $\mathscr{H}'$ by

$$\rho'(\boldsymbol{\theta}) = \sum_k P_k |\mathfrak{P}, \varphi_k\rangle\langle\mathfrak{P}, \varphi_k| \tag{2.22}$$

The probability that the measurement of the operator-valued measure $\{X(\Delta)\}$ yields values of the estimates $\hat{\boldsymbol{\theta}}$ in a region Δ of $\mathfrak{P}$ is now, by equation (2.17),

$$\begin{aligned}\Pr\{\Delta|\boldsymbol{\theta}\} &= \mathrm{Tr}\,[\rho(\boldsymbol{\theta})\,X(\Delta)] = \sum_k P_k\langle\varphi_k|X(\Delta)|\varphi_k\rangle \\ &= \sum_k P_k\langle\mathfrak{P}, \varphi_k|\Delta, \varphi_k\rangle' = \sum_k P_k\langle\mathfrak{P}, \varphi_k|E(\Delta)|\mathfrak{P}, \varphi_k\rangle' \\ &= \mathrm{Tr}'\,[\rho'(\boldsymbol{\theta})\,E(\Delta)]\end{aligned} \tag{2.23}$$

where Tr$'$ is a trace over the space $\mathscr{H}'$, for by equation (2.20)

$$E(\Delta)|\mathfrak{P}, \varphi_k\rangle = |\Delta, \varphi_k\rangle \tag{2.24}$$

Thus if the operators Θ_j', $j = 1, 2, \ldots, m$, defined by equation (2.16) in terms of the projection-valued measure $\{E(\Delta)\}$, are measured in $\mathscr{H}'$, the outcomes $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m$ will have the same probability density function as when the operator-valued measure $\{X(\Delta)\}$ is measured in the original Hilbert space $\mathscr{H}_S$.

The extended space so constructed is not the only possible Hilbert space $\mathscr{H}'$ in which a projection-valued measure $\{E(\Delta)\}$ related to $\{X(\Delta)\}$ by an equation like (2.15) exists. According to Holevo (1972), if the extended space $\mathscr{H}'$ has a large enough dimensionality, it can be represented as the product $\mathscr{H}_S \otimes \mathscr{H}_A$ of the space $\mathscr{H}_S$ and an auxiliary Hilbert space $\mathscr{H}_A$, and in $\mathscr{H}_A$ a density operator ρ_A representing a pure state

$$\rho_A = |\psi_A\rangle\langle\psi_A|$$

can be found such that the density operator in $\mathscr{H}_S \otimes \mathscr{H}_A$ has the product form $\rho(\theta) \otimes \rho_A$. How to determine the appropriate space $\mathscr{H}_A$ and the state $|\psi_A\rangle$ in general is unclear. An example in which the Neumark extension is known is given in the next section.

3. *Adjoined Harmonic Oscillators*

The coherent states $|\alpha\rangle$ of the harmonic oscillator are overcomplete in the sense that

$$\int |\alpha\rangle\langle\alpha|\, d^2\alpha/\pi = \mathbf{1} \tag{3.1}$$

where $\alpha = \xi + i\eta$, $d^2\alpha = d\xi\, d\eta$, and the integral is taken over the entire complex α-plane (Glauber, 1963). We can thus define an operator-valued measure $\{X(\Delta)\}$ by

$$X(\Delta) = \int_{\Delta} |\alpha\rangle\langle\alpha|\, d^2\alpha/\pi \tag{3.2}$$

where Δ is an arbitrary region of the α-plane. This measure satisfies the rules (2.4–2.6), and the operators $X(\Delta)$ and $X(\Delta')$ for disjoint regions Δ and Δ' do not commute. We can define the differential operator by

$$X(\xi, \eta; d\xi\, d\eta) = |\xi + i\eta\rangle\langle\xi + i\eta|\, d\xi\, d\eta/\pi \tag{3.3}$$

The states $|\alpha\rangle$ are the right eigenstates of the annihilation operator $\boldsymbol{a}$, whose commutator with the adjoint creation operator a^+ is as usual

$$[a, a^+] = aa^+ - a^+a = \mathbf{1} \tag{3.4}$$

We define the dimensionless coordinate and momentum operators Q and P by

$$Q = \tfrac{1}{2}(a^+ + a), \qquad P = \tfrac{1}{2}i(a^+ - a); \tag{3.5}$$

their commutator is

$$[Q, P] = i/2 \tag{3.6}$$

The product of the space $\mathscr{H}_S$ of this harmonic oscillator and the space $\mathscr{H}_A$ of an auxiliary harmonic oscillator A forms a Hilbert space $\mathscr{H}' = \mathscr{H}_S \otimes \mathscr{H}_A$ in which the operator-valued measure $\{X(\Delta)\}$ can be extended to a commuting projection-valued measure. The product space is spanned by the continuous simultaneous eigenstates $|\xi, \eta\rangle$ of the commuting operators $Q - Q'$ and $P + P'$, where Q' and P' are the coordinate and momentum operators, defined as in (3.5), for the auxiliary oscillator and

$$a' = Q' + iP' \tag{3.7}$$

is its annihilation operator.

These states can be written as

$$\begin{aligned} |\xi, \eta\rangle &= \pi^{-1/2} \int e^{2i\eta q} |q\rangle_1 |q - \xi\rangle_2\, dq \\ &= \pi^{-1/2} \int e^{2i(\eta - p)\xi} |p\rangle_1 |\eta - p\rangle_2\, dp \end{aligned} \tag{3.8}$$

in terms of the eigenstates $|q\rangle_1 |q'\rangle_2$ of the coordinate operators Q and Q' and the eigenstates $|p\rangle_1 |p'\rangle_2$ of the momentum operators P and P'; the subscripts 1 and 2 refer to the two oscillators. Indeed,

$$\begin{aligned} (Q - Q')|\xi, \eta\rangle &= \pi^{-1/2} \int e^{2i\eta q} [q - (q - \xi)] |q\rangle_1 |q - \xi\rangle_2\, dq \\ &= \xi |\xi, \eta\rangle \end{aligned} \tag{3.9}$$

and

$$(P + P')|\xi,\eta\rangle = \pi^{-1/2}\int e^{2i(\eta-p)\xi}(p+\eta-p)|p\rangle_1|\eta-p\rangle_2\,dp$$
$$= \eta|\xi,\eta\rangle \tag{3.10}$$

These states are normalized so that

$$\langle\xi',\eta'|\xi,\eta\rangle = \delta(\xi-\xi')\,\delta(\eta-\eta') \tag{3.11}$$

The projection of the state $|\xi,\eta\rangle$ onto the ground state $|0\rangle_2$ of the second oscillator yields the state

$$_2\langle 0|\xi,\eta\rangle|0\rangle_2 = \pi^{-1/2}\int e^{2i\eta q}|q\rangle_1\,{}_2\langle 0|q-\xi\rangle_2\,dq\,|0\rangle_2$$
$$= \pi^{-1/2}\,e^{i\eta\xi}|\xi+i\eta\rangle_1|0\rangle_2 \tag{3.12}$$

where $|\xi+i\eta\rangle_1$ is the coherent state for the first oscillator with parameter $\alpha=\xi+i\eta$. The proof is given in the appendix. Thus the projection operator $\int_\Delta |\xi,\eta\rangle\langle\xi,\eta|\,d\xi\,d\eta$ is the Neumark extension of the operator

$$\pi^{-1}\int_\Delta |\alpha\rangle_1\,{}_1\langle\alpha|\,d^2\alpha \otimes |0\rangle_2\,{}_2\langle 0|, \qquad \alpha=\xi+i\eta$$

in the product $\mathscr{H}_S \otimes \mathscr{H}_A$ of the Hilbert spaces of the two oscillators. The former provides a projection-valued measure in $\mathscr{H}_S \otimes \mathscr{H}_A$, the latter an operator-valued measure in the subspace spanned by $|w_k\rangle_1|0\rangle_2$, where the $|w_k\rangle_1$ form an arbitrary complete orthonormal set in $\mathscr{H}_S$, and $|0\rangle_2$ is the ground state of the second oscillator. The outcomes ξ and η of measuring in $\mathscr{H}_S$ the noncommuting operator-valued measure $\{X(\Delta)\}$ defined by equation (3.2) have the same joint p.d.f. as the outcomes of measuring the commuting operators $Q-Q'$ and $P+P'$ in $\mathscr{H}_S \otimes \mathscr{H}_A$ (Arthurs & Kelly, 1965; Personick, 1971a). The ideal simultaneous measurement of noncommuting observables discussed by She & Heffner (1966) can be regarded as measurement of this operator-valued measure.

4. *The Cramér–Rao Inequalities*

Let an operator-valued measure $\{X(\Delta)\}$ be generated by the infinitesimal operators $X(\boldsymbol{\theta}';d^m\boldsymbol{\theta}')$ as in equation (2.7). The conditional expected value of the estimate of the parameter θ_j when $\{X(\Delta)\}$ is measured is

$$\mathbf{E}(\hat{\theta}_j|\boldsymbol{\theta}) = \bar{\theta}_j = \int_{\mathfrak{P}} \theta_j'\,\mathrm{Tr}_S\,\rho(\boldsymbol{\theta})\,X(\boldsymbol{\theta}';d^m\boldsymbol{\theta}') \tag{4.1}$$

the true values of the parameters being $\boldsymbol{\theta}$. If

$$\bar{\theta}_j = \theta_j \tag{4.2}$$

the estimate of θ_j is said to be unbiased. We deal here for simplicity only with unbiased estimators. The covariances of the errors are similarly defined by

$$B_{ij} = \mathbf{E}[(\hat{\theta}_i-\bar{\theta}_i)(\hat{\theta}_j-\bar{\theta}_j)|\boldsymbol{\theta}]$$
$$= \mathrm{Tr}_S\int_{\mathfrak{P}} (\theta_i'-\bar{\theta}_i)(\theta_j'-\bar{\theta}_j)\,\rho(\boldsymbol{\theta})\,X(\boldsymbol{\theta}';d^m\boldsymbol{\theta}') \tag{4.3}$$

Because the operators $X(\boldsymbol{\theta}';d^m\boldsymbol{\theta}')$ are positive-definite, so is the matrix $\mathbf{B} = \|B_{ij}\|$. In the sequel we write simply Tr for Tr_S.

We can derive a Cramér–Rao inequality for unbiased estimators by following a procedure quite similar to the derivation in classical statistics. Differentiating equation (4.1) with respect to θ_k and using equation (4.2), we get

$$\mathrm{Tr} \int_{\mathfrak{P}} \theta_j' \frac{\partial \rho}{\partial \theta_k} X(\boldsymbol{\theta}';d^m\boldsymbol{\theta}') = \delta_{jk} \tag{4.4}$$

By equation (2.6), and because $\mathrm{Tr}\,\rho(\boldsymbol{\theta}) \equiv \mathbf{1}$,

$$\mathrm{Tr} \int_{\mathfrak{P}} \frac{\partial \rho}{\partial \theta_k} X(\boldsymbol{\theta}';d^m\boldsymbol{\theta}') = 0 \tag{4.5}$$

Multiplying this by θ_j, subtracting from equation (4.4), and introducing the right logarithmic-derivative (r.l.d.) operators L_k by

$$\partial\rho/\partial\theta_k = \rho L_k = L_k^+ \rho \tag{4.6}$$

we get

$$\mathrm{Tr} \int_{\mathfrak{P}} (\theta_j' - \theta_j) L_k^+ \rho X(\boldsymbol{\theta}';d^m\boldsymbol{\theta}') = \delta_{jk} \tag{4.7}$$

This we multiply by $y_j^+ z_k$ and sum over j and k to obtain

$$\begin{aligned} \mathbf{Y}^+\mathbf{Z} &= \sum_{j=1}^{m} y_j^* z_j \\ &= \mathrm{Tr} \sum_{j=1}^{m} \sum_{k=1}^{m} \int_{\mathfrak{P}} y_j^*(\theta_j' - \theta_j) z_k L_k^+ \rho X(\boldsymbol{\theta}';d^m\boldsymbol{\theta}') \\ &= \mathrm{Tr} \int_{\mathfrak{P}} \eta^* \rho^{1/2} X(\boldsymbol{\theta}';d^m\boldsymbol{\theta}') \zeta \rho^{1/2} \end{aligned} \tag{4.8}$$

where $\mathbf{Y}^+ = (y_1^*, y_2^*, \ldots, y_m^*)$ is a row vector of arbitrary complex elements y_j^*, $\mathbf{Z}$ is a column vector of arbitrary complex elements z_j, and

$$\begin{aligned} \eta &= \sum_{j=1}^{m} y_j(\theta_j' - \theta_j) \\ \zeta &= \sum_{k=1}^{m} L_k^+ z_k \end{aligned} \tag{4.9}$$

Now we employ a version of the Schwarz inequality, which depends on the positive definiteness of the operators X; for any operators $\mathbf{O}_1$ and $\mathbf{O}_2$, which may depend on $\boldsymbol{\theta}'$, and for any complex number λ,

$$\mathrm{Tr} \int_{\mathfrak{P}} (\mathbf{O}_1 - \lambda \mathbf{O}_2) X(\boldsymbol{\theta}';d^m\boldsymbol{\theta}') (\mathbf{O}_1^+ - \lambda^* \mathbf{O}_2^+) \geqslant 0 \tag{4.10}$$

and by minimizing in the usual way with respect to λ, we find the inequality

$$\left[\operatorname{Tr}\int_{\mathfrak{P}} \mathbf{O}_1 X(\theta'; d^m\theta')\,\mathbf{O}_1^+\right]\left[\operatorname{Tr}\int_{\mathfrak{P}} \mathbf{O}_2 X(\theta'; d^m\theta')\,\mathbf{O}_2^+\right] \geqslant \left|\operatorname{Tr}\int_{\mathfrak{P}} \mathbf{O}_2 X(\theta'; d^m\theta')\,\mathbf{O}_1^+\right|^2 \tag{4.11}$$

Here we take

$$\mathbf{O}_1 = \rho^{1/2}\zeta^+, \qquad \mathbf{O}_2 = \eta^*\rho^{1/2} \tag{4.12}$$

and obtain from equations (4.8) and (4.11)

$$\begin{aligned}|\mathbf{Y}^+\mathbf{Z}|^2 &\leqslant \left[\operatorname{Tr}\int_{\mathfrak{P}} \rho^{1/2}\zeta^+ X(\theta'; d^m\theta')\,\zeta\rho^{1/2}\right] \\ &\quad\times\left[\operatorname{Tr}\int_{\mathfrak{P}} \eta^*\rho^{1/2} X(\theta'; d^m\theta')\,\rho^{1/2}\eta\right] \\ &= \left[\operatorname{Tr}\int_{\mathfrak{P}} \zeta\rho\zeta^+ X(\theta'; d^m\theta')\right]\left[\operatorname{Tr}\int_{\mathfrak{P}} |\eta|^2\rho X(\theta'; d^m\theta')\right]\end{aligned} \tag{4.13}$$

Since ρ and ζ do not depend on $\boldsymbol{\theta}'$, we can integrate over the parameter space $\mathfrak{P}$ in the first bracket on the right by using equation (2.6). Thus we finally obtain the inequality

$$|\mathbf{Y}^+\mathbf{Z}|^2 \leqslant (\mathbf{Y}^+\mathbf{B}\mathbf{Y})(\mathbf{Z}^+\mathbf{A}\mathbf{Z}) \tag{4.14}$$

where $\mathbf{B}$ is the covariance matrix defined in equation (4.3) and $\mathbf{A} = \|A_{ij}\|$ has the matrix elements

$$A_{ij} = \operatorname{Tr}(\rho L_i L_j^+) \tag{4.15}$$

If in particular we put

$$\mathbf{Z} = \mathbf{A}^{-1}\mathbf{Y} \tag{4.16}$$

we obtain

$$\mathbf{Y}^+\mathbf{B}\mathbf{Y} \geqslant \mathbf{Y}^+\mathbf{A}^{-1}\mathbf{Y} \tag{4.17}$$

from which inequalities for various combinations of the variances and covariances of the unbiased estimates can be obtained by appropriate choices of the column vector $\mathbf{Y}$. Furthermore, by expressing the positive-definite matrix $\mathbf{G} = \|g_{ij}\|$ in terms of its eigenvectors and eigenvalues, we can bound the expected value of the quadratic cost function in equation (1.4) by

$$\mathbf{E}[C(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})|\boldsymbol{\theta}] = \operatorname{Tr}\mathbf{G}\mathbf{B} \geqslant \operatorname{Tr}\mathbf{G}\mathbf{A}^{-1} \tag{4.18}$$

From what has been said in Section 2, these bounds apply as well to estimates based on measurements of commuting operators on a combination of the system S with an auxiliary apparatus A, when the density operator has the product form $\rho(\boldsymbol{\theta}) \otimes \rho_A$, with ρ_A independent of the unknown parameters $\boldsymbol{\theta}$. They also cover estimates based on the outcomes of a temporal succession of measurements on the system.

Equality holds in equation (4.11) if the operators $X(\boldsymbol{\theta}'; d^m\boldsymbol{\theta}')$ obey

$$(\mathbf{O}_1 - \lambda\mathbf{O}_2)\, X(\boldsymbol{\theta}'; d^m\boldsymbol{\theta}') = \mathbf{0}$$

for some complex number λ. Translating by means of equations (4.12), (4.9), and (4.16), we require for some λ

$$\sum_{j=1}^{m}\sum_{k=1}^{m} y_j^*(\mathbf{A}^{-1})_{jk}\,\rho L_k\, X(\boldsymbol{\theta}'; d^m\boldsymbol{\theta}') = \lambda\rho \sum_{j=1}^{m} y_j^*(\theta_j' - \theta_j)\, X(\boldsymbol{\theta}'; d^m\boldsymbol{\theta}') \quad (4.19)$$

An example will be presented in Section 5.

A second Cramér–Rao inequality can be based on the symmetrized logarithmic derivative (s.l.d.) operators $\mathscr{L}_j$ defined by

$$\partial\rho/\partial\theta_j = \tfrac{1}{2}(\rho\mathscr{L}_j + \mathscr{L}_j^+\rho) \quad (4.20)$$

and by adapting the analysis in a previous paper (Helstrom, 1968a) to this new format, we can show that

$$\tilde{\mathbf{Y}}\mathbf{B}\mathbf{Y} > \tilde{\mathbf{Y}}\mathscr{A}^{-1}\mathbf{Y} \quad (4.21)$$

where now the elements of the column vector $\mathbf{Y}$ and its transposed row vector $\tilde{\mathbf{Y}} = (y_1, y_2, \ldots, y_m)$ must be real numbers, and the elements of the symmetric matrix $\mathscr{A}$ are given by

$$\mathscr{A}_{ij} = \tfrac{1}{2}\operatorname{Tr}\rho(\mathscr{L}_i\mathscr{L}_j^+ + \mathscr{L}_j\mathscr{L}_i^+) \quad (4.22)$$

Equality in (4.21) requires

$$\rho\sum_{i=1}^{m}\sum_{j=1}^{m} y_i(\mathscr{A}^{-1})_{ij}\mathscr{L}_j\, X(\boldsymbol{\theta}'; d^m\boldsymbol{\theta}') = \lambda\rho\sum_{j=1}^{m} y_i(\theta_i' - \theta_i)\, X(\boldsymbol{\theta}'; d^m\boldsymbol{\theta}') \quad (4.23)$$

5. *Estimating the Complex Amplitude of a Harmonic Oscillator*

The density operator of a simple harmonic oscillator in thermal equilibrium with a heat bath at absolute temperature $\mathscr{T}$ and containing a coherent oscillation of complex amplitude $\mu = \mu_1 + i\mu_2$ is given in the P-representation by

$$\rho(\mu_1, \mu_2) = (\pi N)^{-1}\int \exp(-|\alpha - \mu|^2/N)\,|\alpha\rangle\langle\alpha|\, d^2\alpha \quad (5.1)$$

(Glauber, 1963), where the $|\alpha\rangle$ are coherent states as in Section 3 and

$$N = (e^{h\nu/K\mathscr{T}} - 1)^{-1}$$

is the mean number of thermal photons, $h\nu$ being the quantum of energy and K Boltzmann's constant. The oscillator might represent a mode of an ideal receiver of a coherent electromagnetic signal (Helstrom, 1972). The mean number of photons attributable to the signal in the mode is $N_s = |\mu|^2$.

To be estimated are the real and imaginary parts of the complex amplitude μ.

The r.l.d. operators are now

$$L_1 = \frac{a^+ - \mu^*}{N} + \frac{a - \mu}{N+1}$$
$$L_2 = i\left(\frac{a^+ - \mu^*}{N} - \frac{a-\mu}{N+1}\right) \tag{5.2}$$

in terms of the annihilation and creation operators a and a^+ of the mode field, as can be shown by equation (3.20) of Helstrom (1967b), and the s.l.d. operators are

$$\mathscr{L}_1 = 4(Q - m_1)/(2N+1)$$
$$\mathscr{L}_2 = 4(P - m_2)/(2N+1) \tag{5.3}$$

with Q and P defined by equation (3.5) (Helstrom, 1968a). The matrices $\mathbf{A}^{-1}$ and $\mathscr{A}^{-1}$ defined through equations (4.15) and (4.22) are now

$$\mathbf{A}^{-1} = \tfrac{1}{4}\begin{bmatrix} 2N+1 & i \\ -i & 2N+1 \end{bmatrix} \tag{5.4}$$

$$\mathscr{A}^{-1} = \tfrac{1}{4}\begin{bmatrix} 2N+1 & 0 \\ 0 & 2N+1 \end{bmatrix} \tag{5.5}$$

The vectors $\tilde{\mathbf{Y}} = (1\ 0)$ and $(0\ 1)$ give with each form of the Cramér–Rao inequality the bounds

$$\operatorname{Var}\hat{\mu}_i \geqslant \tfrac{1}{4}(2N+1), \qquad i = 1, 2 \tag{5.6}$$

and equality is obtained for $\operatorname{Var}\hat{\mu}_1$ by measuring the estimating operator Q and for $\operatorname{Var}\hat{\mu}_2$ by measuring P. These cannot both be measured simultaneously on the same system.

If we put $\mathbf{Y}^+ = (1\ i)$ into equation (4.17), we obtain an inequality due to Yuen & Lax (1972),

$$\operatorname{Var}\hat{\mu}_1 + \operatorname{Var}\hat{\mu}_2 \geqslant N+1 \tag{5.7}$$

and equation (4.19) states after multiplication by ρ^{-1}, which exists,

$$(a-\mu)\, X(\mu_1', \mu_2'; d\mu_1'\, d\mu_2') = \lambda(\mu' - \mu)\, X(\mu_1', \mu_2'; d\mu_1'\, d\mu_2') \tag{5.8}$$

This is satisfied with $\lambda = 1$ by

$$X(\mu_1', \mu_2'; d\mu_1'\, d\mu_2') = |\mu'\rangle\langle\mu'|\, d\mu_1'\, d\mu_2'/\pi \tag{5.9}$$

where $|\mu'\rangle$ is the coherent state of complex amplitude $\mu' = \mu_1' + i\mu_2'$, which is the right eigenstate of the annihilation operator a (Glauber, 1963). Thus the lower bound $N+1$ on the sum of the variances of unbiased estimates of the real and imaginary parts, μ_1 and μ_2, of the complex amplitude μ of the signal is attained by a device that measures the operator-valued measure described in Section 3. As shown there, that procedure is equivalent to measuring the commuting operators $Q - Q'$ and $P + P'$ on the combination of the original harmonic oscillator with an auxiliary one in the ground state (Personick, 1971a).

When $N = 0$, the density operator in equation (5.1) reduces to that of a pure state $|\mu\rangle\langle\mu|$ representing a minimum-uncertainty wave packet. Equation (5.6) then is consistent with the usual form of the uncertainty principle,

$$(\mathrm{Var}\,\hat{\mu}_1\,\mathrm{Var}\,\hat{\mu}_2)^{1/2} \geqslant \tfrac{1}{4} \tag{5.10}$$

and refers to estimates of μ_1 and μ_2 on different systems. The estimates obtained by measuring $Q - Q'$ and $P + P'$ as stated above yield, however,

$$(\mathrm{Var}\,\hat{\mu}_1\,\mathrm{Var}\,\hat{\mu}_2)^{1/2} = \tfrac{1}{2} \tag{5.11}$$

consistently with the results of Arthurs & Kelly (1965).

The two quantum-mechanical forms of the Cramér–Rao inequality have been applied by Helstrom & Kennedy (1972) to the estimation of the arrival time τ and the angular carrier frequency Ω of a narrow-band, coherent pulse signal received in the presence of thermal noise. Superior bounds on the minimum variances of individual estimates of τ or Ω are given by the inequality, equation (4.21), based on the s.l.d. operators $\mathscr{L}_j$,

$$\begin{aligned} \mathrm{Var}\,\hat{\tau} &\geqslant (D^2\,\Delta\omega^2)^{-1} \\ \mathrm{Var}\,\hat{\Omega} &\geqslant (D^2\,\Delta t^2)^{-1} \end{aligned} \tag{5.12}$$

where

$$D^2 = 4N_s/(2N+1) \tag{5.13}$$

is a signal-to-noise ratio, with N given in equation (5.2) and N_s the mean number of photons borne by the signal; Δt^2 and $\Delta\omega^2$ are mean-square time and angular-frequency widths of the pulse (Helstrom, 1968b, p. 18 ff). When $N \gg 1$ these reduce to the lower bounds set by classical estimation theory (Helstrom, 1968b, p. 282). (These results are for the sake of simplicity given for a purely amplitude-modulated pulse.) For simultaneous estimation of both arrival time τ and carrier frequency Ω, on the other hand, the inequality (4.17) yields the superior lower bound on a weighted sum of squared errors

$$R = \frac{\mathrm{Var}\,\hat{\tau}}{\Delta t^2} + \frac{\mathrm{Var}\,\hat{\Omega}}{\Delta\omega^2} \geqslant \frac{2N(N+1)}{N_s\,\Delta\omega\,\Delta t\,[(2N+1)\,\Delta\omega\,\Delta t - \frac{1}{2}]} \tag{5.14}$$

when $\frac{1}{2} \leqslant \Delta\omega\,\Delta t \leqslant N + \frac{1}{2}$. The minimum value of the product $\Delta\omega\,\Delta t$ is $\frac{1}{2}$, attained for a pulse with a Gaussian form. For $\Delta\omega\,\Delta t > N + \frac{1}{2}$, on the other hand, equation (4.21) yields the superior bound,

$$R \geqslant \frac{2N+1}{2N_s\,\Delta\omega^2\,\Delta t^2}, \qquad \Delta\omega\,\Delta t > N + \tfrac{1}{2} \tag{5.15}$$

For $N \gg 1$ both yield the classical result obtained by weighting and adding the inequalities in (5.12). In the domain of quantum-theory, however, the lower limit on the risk R for joint measurement of the arrival time and carrier frequency of a signal in the same receiver may be greater than the lower limit on R when those parameters are measured on signals in separate receivers.

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Appendix: Proof of Equation (3.12)

$$\begin{aligned} {}_2\langle 0|\xi,\eta\rangle &= \pi^{-1/2}\int e^{2i\eta q}|q\rangle_1\,{}_2\langle 0|q-\xi\rangle_2\,dq \\ &= (2/\pi)^{1/4}\pi^{-1/2}\int e^{2i\eta q-(q-\xi)^2}|q\rangle_1\,dq \\ &= (2/\pi)^{1/4}\pi^{-1/2}e^{-\xi^2}\int e^{-q^2+2\alpha q}|q\rangle_1\,dq \end{aligned}$$

with $\alpha = \xi + i\eta$, where we have used

$${}_2\langle q|\beta\rangle = (2/\pi)^{1/4}\exp[-(q^2-2\beta q+\tfrac{1}{2}|\beta|^2+\tfrac{1}{2}\beta^2)]$$

for the second oscillator, with $\beta = 0$ (Glauber, 1963). However, for the first oscillator, the coherent state $|\alpha\rangle$ is

$$\begin{aligned} |\alpha\rangle &= \int |q\rangle_1\,{}_1\langle q|\alpha\rangle\,dq \\ &= (2/\pi)^{1/4}\int |q\rangle_1\exp-[q^2-2\alpha q+\tfrac{1}{2}|\alpha|^2+\tfrac{1}{2}\alpha^2]\,dq \end{aligned}$$

with which our first equation yields immediately

$${}_2\langle 0|\xi,\eta\rangle = \pi^{-1/2}e^{i\xi\eta}|\alpha\rangle$$

whence equation (3.12) follows.

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